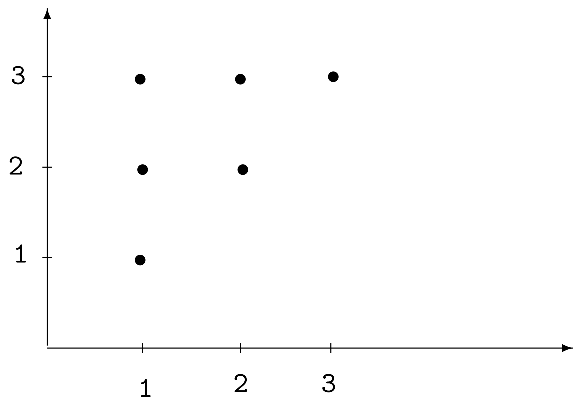
CHAPTER 7

Relations and Functions

7.1 ANSWER: a) $\mathcal{R} = \{(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)\}$.



b) The same figure, but with the points below and on the diagonal.

7.2 ANSWER: $\mathcal{R} = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(1,7),(1,8),(1,9),(1,10),$
$(2,2),(2,4),(2,6),(2,8),(2,10),(3,3),(3,6),(3,9),(4,4),(4,8),(5,5),(5,10),$
$(6,6),(7,7),(8,8),(9,9)(10,10)\}$.

7.3 ANSWER: $\mathcal{R} = \{(2,3),(2,5),(2,7),(3,2),(3,4),(3,5),(3,7),(4,3),(4,5),(4,7),$
$(5,2),(5,3),(5,4),(5,6),(5,7),(6,5),(6,7),(7,2),(7,3),(7,4),(7,5),(7,6)\}$.

7.4 ANSWER: $\mathcal{R} = \{(1,1),(1,4),(1,7),(1,10),(2,2),(2,5),(2,8),(3,3),(3,6),(3,9),$
$(4,1),(4,4),(4,7),(4,10),(5,2),(5,5),(5,8),(6,3),(6,6),(6,9),(7,1),(7,4),$
$(7,7),(7,10),(8,2),(8,5),(8,8),(9,3),(9,6),(9,9),(10,1),(10,4),(10,7),$
$(10,10)\}$.

7.5 ANSWER: $\mathcal{R} = \{(-2,-2),(-2,2),(-1,-1),(-1,1),(0,0),(1,-1),(1,1),(2,-2),$
$(2,2)\}$.

7.6 ANSWER: $\mathcal{R} = \{(1,-2),(1,-1),(1,0),(1,1),(1,2),(-2,-2),(-2,2),(-1,-1),$ $(-1,1),\ (0,0),\ (2,-2),\ (2,2)\}$.

7.7 ANSWER: $\mathcal{R} = \{(1,1),(1,3),(2,2),(2,4),(3,1),(3,3),(4,2),(4,4)\}$.

7.9 a) Not reflexive, since $2\mathcal{R}2$ does not hold. Symmetric, since $x\mathcal{R}y \rightarrow y\mathcal{R}x$ holds for each x, y (even though $1\mathcal{R}2$ and $2\mathcal{R}1$ do not hold, remember that an implication is automatically true if the antecedent is false). Transitive, since the only possibility that the antecedent is true in the implication $(x\mathcal{R}y \wedge y\mathcal{R}z) \rightarrow x\mathcal{R}z$ is that x, y, z are 1 or 3, and then it holds that $x\mathcal{R}z$ since all relations hold between these elements.

b) Reflexive, since $x\mathcal{R}x$ holds for all x. Symmetric as in a). Transitive.

c) Reflexive, since $x\mathcal{R}x$ holds for all x. Not symmetric since $1\mathcal{R}2$ holds but not $2\mathcal{R}1$. Not transitive, since $3\mathcal{R}1$ and $1\mathcal{R}2$ but not $3\mathcal{R}2$.

d) Reflexive, not transitive, symmetric. Clue: $x\mathcal{R}y$ means that x is at most 1 distant from y on the real line. That the relation is not transitive is clear from instances $1\mathcal{R}2$ and $2\mathcal{R}3$ but not $1\mathcal{R}3$.

7.10 a) The function yields $f(1) = 4, f(2) = 5, f(3) = 3, f(4) = 1, f(5) = 2$, so the relation becomes $\mathcal{R} = \{(1,4),(2,5),(3,3),(4,1),(5,2)\}$.

b) Not reflexive, symmetric, not transitive (since for instance $1\mathcal{R}4$ and $4\mathcal{R}1$ but not $1\mathcal{R}1$).

7.11 Yes, 7.9b.

7.12 a) Yes. b) No (since $\mathcal{R}$ is not reflexive). c) Yes. d) No (since not symmetric). e) No (since not symmetric). f) Yes. g) Yes.

7.13 Yes.

7.14 Yes.

7.15 a) Partition. b) Not a partition, since A_2 and A_3 are not disjunct. c) Not a partition, since, since all subsets must be non-empty in order for the subsets may constitute a partition. d) Partition. e) Not a partition, since element t does not belong to any subset.

7.16 a) $\{\{1,3\},\{2,4\}\}$, since $1\mathcal{R}3$ and $2\mathcal{R}4$ and $1\mathcal{R}1, 3\mathcal{R}3, 2\mathcal{R}2, 4\mathcal{R}4$.

Construct a partition from the given set by dividing the set into areas (rather like drawing a map) by placing in each area those elements that are related to one another. This is rather like dividing all cars into groups according to their colour, so that red cars end up in one group and blue cars in another and so on. Place those elements together that pair wise fulfill the property that their difference is an even number. The numbers 1 and 3 yield differences of 2, -2 and 0 (the difference with themselves) which are even numbers. No more numbers in the set A yields an even difference, either with 1 or 3, so $\{1,3\}$ becomes an equivalence class. In the same way numbers 2 and 4 yield differences 2, -2 and 0. So the second equivalence class becomes $\{2,4\}$. There are no further elements in A, so the partition consists of two subsets and is

$\{\{1,3\},\{2,4\}\}$.
b) $\{\{1,4,7,10\},\{2,5,8\},\{3,6,9\}\}$.
c) $\{\{(1,1)\},\{(1,2),(2,1)\},\{(1,3),(2,2),(3,1)\},\{(2,3),(3,2)\},\{(3,3)\}\}$.
d) The partition is $\{A_0, A_1, A_2, A_3\}$, where $A_0 = \{\varnothing, \{1\}\}$, $A_1 = \{\{1,2\},\{2\}\}$, $A_2 = \{\{1,3\},\{3\}\}$, $A_3 = \{\{2,3\},\{1,2,3\}\}$.
Here the set A consists of 8 elements $\{a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$ where

$$\begin{aligned} a_0 &= \varnothing \\ a_1 &= \{1\} \\ a_2 &= \{2\} \\ a_3 &= \{3\} \\ a_4 &= \{1,2\} \\ a_5 &= \{1,3\} \\ a_6 &= \{2,3\} \\ a_7 &= \{1,2,3\} \end{aligned}$$

Each element in A is itself a subset that consists of element from the set $\{1,2,3\}$. (A is thus a set of sets). Two elements are related to each other if they form the same intersection with the set $B = \{2,3\}$. This means that the only thing that matters concerning whether two of the subsets $a_0, ..., a_7$ are related to one another is how they appear regarding elements 2 and 3, anything outside of $\{2,3\}$ is of no consequence. This means that

$$\begin{gathered} a_0 \mathcal{R} a_1 \\ a_2 \mathcal{R} a_4 \\ a_3 \mathcal{R} a_5 \\ a_6 \mathcal{R} a_7 \end{gathered}$$

Furthermore, every set is always related to itself (since $X \cap B = X \cap B$ always holds). This yields the following equivalence classes:

$$\{a_0, a_1\}, \{a_2, a_4\}, \{a_3, a_5\}, \{a_6, a_7\}.$$

e) The partition consists of the population divided into all non-empty age groups.
f) The partition consists of $\{M, K\}$, where $M =$ the set of all men, $K =$ the set of all women.

7.17 $\mathcal{R} = \{(1,1),(1,3),(3,1),(1,6),(6,1),(3,3),(3,6),(6,3)(6,6),(2,2),(2,5),(5,2),(5,5),(4,4)\}$.

7.18 $\mathcal{R} = \{(1,1),(2,2),(3,3),(3,4),(4,3),(4,4),(5,5)\}$.

7.19 Answer: 7.

7.20 Answer: 6.

7.21 Answer: [3] and [9] are the same, [6] and [−12] are the same.

7.22 ANSWER: 1. Two number have the same final digit if they are congruent modulo 10. Calculating with modulo yields $9^{1998} \equiv (9^2)^{999} \equiv 1^{999} \equiv 1$.

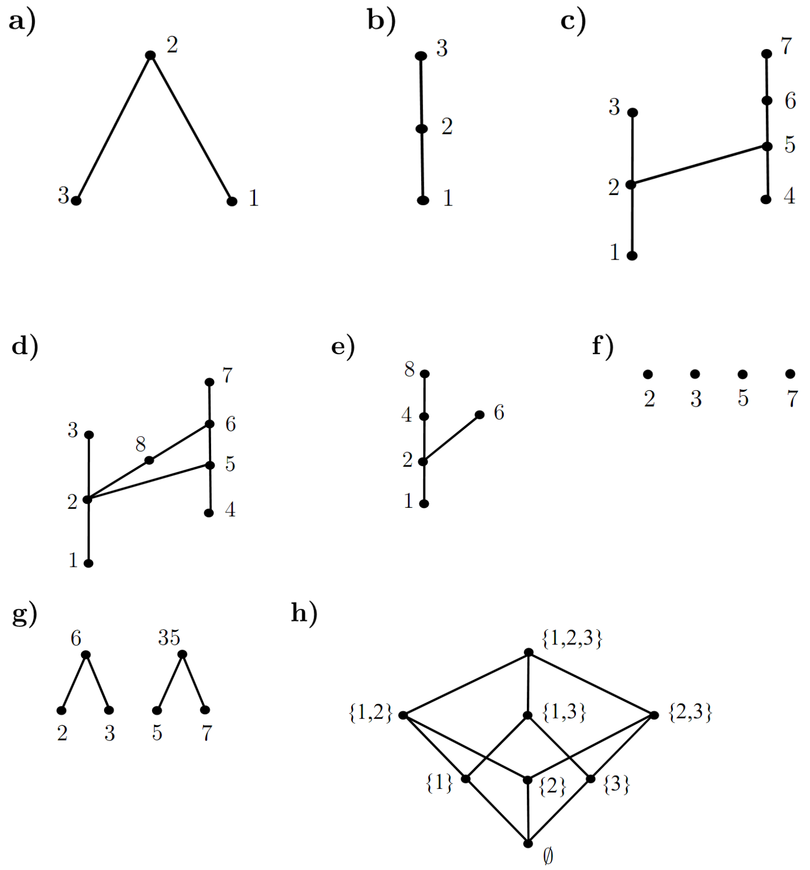
7.23 ANSWER: 6.

7.24 ANSWER: 1. Calculating with congruence modulo 10: $11^{333} \equiv 1^{333} \equiv 1$.

7.25 ANSWER: 2. Calculating with congruence modulo 10: $2^{333} \equiv (2^3)^{111} \equiv (2^9)^{37} \equiv 512^{37} \equiv 2^{37} \equiv 2 \cdot 2^{36} \equiv 2 \cdot (2^9)^4 \equiv 2 \cdot 512^4 \equiv 2 \cdot 2^4 \equiv 32 \equiv 2$.

7.26 a) ANSWER: Yes, since $x\mathcal{R}y \wedge y\mathcal{R}x$ holds only if $x = y = 1$. b) ANSWER: No, since $x\mathcal{R}y \wedge y\mathcal{R}x$ holds for $x = 1, y = 2$ and then $x = y$ does not hold. c) ANSWER: Yes, since $x\mathcal{R}y \wedge y\mathcal{R}x \Longrightarrow x = y$ holds trivially for all x, y since the antecedent in the implication is not true for any x, y.

7.27



7.28 a) $A = \{a, b, c, d\}, \mathcal{R} = \{(a, a), (b, b), (c, c), (d, d), (a, c), (a, d), (b, c), (b, d), (c, d)\}$.
b) $A = \{1, 2, 3, 4\}, \mathcal{R} = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (3, 2), (3, 4)\}$.
c) $A = \{x, 2, \pi\}, \mathcal{R} = \{(x, x), (2, 2), (\pi, \pi), (x, 2), (x, \pi)\}$.
d) $A = \{\{1\}, t, B, C\}, \mathcal{R} = \{(\{1\}, \{1\}), (t, t), (B, B), (C, C), (B, C)\}$.

7.29 a) Min: 1, 3. Max: 2. b) Min: 1. Max: 3. c) Min: 1, 4. Max: 3, 7. d) Min: 1, 4. Max: 3, 7. e) Min: 1. Max: 6, 8. f) Min: 2, 3, 5, 7. Max: 2, 3, 5, 7. g) Min: 2, 3, 5, 7. Max: 6, 35. h) Min: $\varnothing$. Max: $\{1, 2, 3\}$.

7.30 a) a, b, c, d. b) $1, 3, 2, 4$ (or $1, 3, 4, 2$). c) $x, 2, \pi$. d) $\{1\}, t, B, C$ (or for instance $B, C, \{1\}, t$)

7.31 $1, 2, 3, 4, 5, 8, 6, 7$.

7.32 $(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)$.

7.33 a) Function. b) Not a function, since both $1\mathcal{R}2$ and $1\mathcal{R}3$ hold. c) Not a function with the domain X, but it is a function with the domain $X_2 = \{1, 3\}$.

7.34 $\mathcal{V}_f = [1, 9]$. The function is increasing in the interval, so the function takes all values between $f(1) = 1$ and $f(3) = 9$.

7.35 $\mathcal{V}_f = [1, 10)$.

7.36 $\mathcal{D}_f = \mathbb{R} \setminus \{0\}$. $\mathcal{V}_f = \mathbb{R}$.

7.37 $\mathcal{D}_f = (-\infty, 1] = \{x \in \mathbb{R} : x \leq 1\}$. $\mathcal{V}_f = [0, +\infty) = \{x \in \mathbb{R} : x \geq 0\}$.

7.38 a) $g(f(x)) = 3(2x + 3) + 4 = 6x + 13$. b) $g(f(x)) = 2(3x + 4) + 3 = 6x + 11$. c) $g(f(x)) = \dfrac{1}{1/x} = x$. d) $g(f(x)) = x$. e) $g(f(x)) = \dfrac{7x - 1}{2x + 1}$. f) $g(f(x)) = 1 - x^2 + (1 - x^2)^2 = 1 - x^2 + 1 - 2x^2 + x^4 = 2 - 3x^2 + x^4$.

7.39 a) Not surjective but injective. b) Surjective and injective. c) Surjective but not injective. d) Neither surjective not injective. e) Surjective but not injective.

7.40 a) $f^{-1}(x) = x/6$. If $y = 6x$ then $x = y/6$, so $x = f^{-1}(y) = y/6$. b) $f^{-1}(x) = (x-1)/6$. If $y = 6x+1$ then $6x = y-1, x = (y-1)/6$, so $x = f^{-1}(y) = (y-1)/6$. c) $f^{-1}(x) = -x$. If $y = -x$ then $x = -y$. d) The function is not 1–1, so there is no inverse.

7.41 a) Show that all function values are unique. For positive x then $f(x)$ is an odd integer, for non-positive x then $f(x)$ is an even number. So all function values for positive x differ from those for all non-positive x. But all function value for positive x are unique because $f(x)$ is an increasing function for positive x. Similarly, all function values for non-positive x are unique because $f(x)$ is a decreasing function for these x. Thus all function values are unique.
b) Show that every natural number is a function value for some $x \in \mathbb{Z}$. This can be seen by observing that when x varies of the positive integers, then $f(x)$ varies over all odd positive integers, and when x varies over the non-positive integers, then $f(x)$ varies of all even positive integers.
c) Let $y = f(x)$. If $y \in \mathbb{Z}^+$ is an even number then $y = -2x$ for some $x \leq 0$, i.e. $f^{-1}(y) = -\dfrac{y}{2}$. If $y \in \mathbb{Z}^+$ is an even number then $y = 2x-1$ for some $x > 0$,

i.e. $f^{-1}(y) = \dfrac{y+1}{2}$. So it holds that

$$f^{-1}(y) = \begin{cases} \dfrac{y+1}{2}, \text{ if } y \text{ is odd} \\ \\ -\dfrac{y}{2}, \text{ if } y \text{ is even.} \end{cases}$$

In order to change variable so that x is a the running variable in f^{-1} too (this can be of interest for example when studying f and f^{-1} in the same coordinate system) then this yields

$$f^{-1}(x) = \begin{cases} \dfrac{x+1}{2}, \text{ if } x \text{ is odd} \\ \\ -\dfrac{x}{2}, \text{ if } x \text{ is even.} \end{cases}$$