

CHAPTER 5

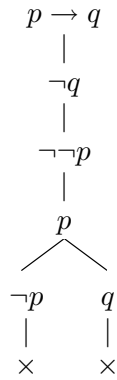
Rule Systems

5.1 $\neg\neg p \vdash_t p$

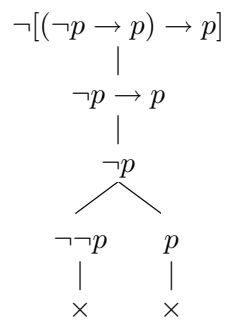
$$\begin{array}{c} \neg\neg p \\ | \\ \neg p \\ | \\ \times \end{array}$$
5.2 $p \vdash_t \neg\neg p$

$$\begin{array}{c} p \\ | \\ \neg\neg p \\ | \\ \neg p \\ | \\ \times \end{array}$$

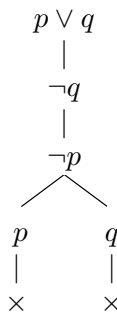
5.3 $p \rightarrow q, \neg q \vdash_t \neg p$



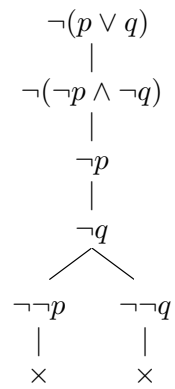
5.4 $\vdash_t (\neg p \rightarrow p) \rightarrow p$



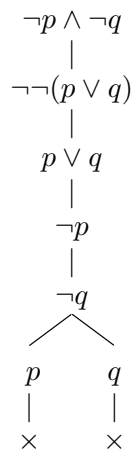
5.5 $p \vee q, \neg q \vdash_t p$



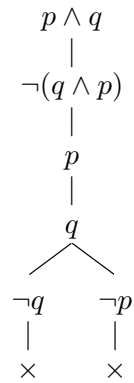
5.6 $\neg(p \vee q) \vdash_t \neg p \wedge \neg q$



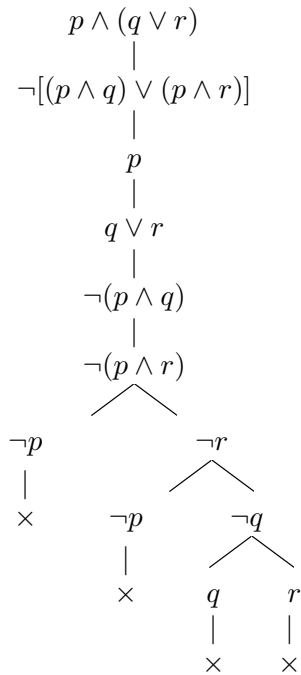
5.7 $\neg p \wedge \neg q \vdash_t \neg(p \vee q)$



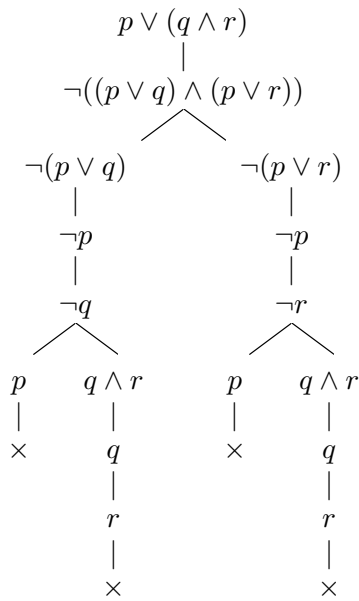
5.8 $p \wedge q \vdash_t q \wedge p$



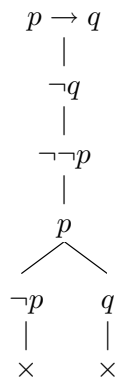
5.9 $p \wedge (q \vee r) \vdash_t (p \wedge q) \vee (p \wedge r)$



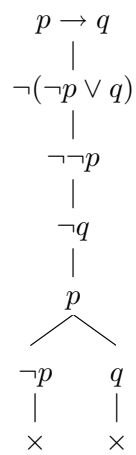
5.10 $p \vee (q \wedge r) \vdash_t (p \vee q) \wedge (p \vee r)$



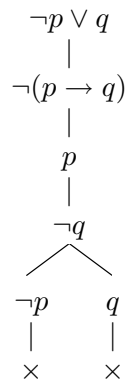
5.11 $p \rightarrow q, \neg q \vdash_t \neg p$



5.12 $p \rightarrow q \vdash_t \neg p \vee q$



5.13 $\neg p \vee q \vdash_t p \rightarrow q$



5.14 $\vdash_t p \vee \neg p$

$$\begin{array}{c} \neg(p \vee \neg p) \\ | \\ \neg p \\ | \\ \neg\neg p \\ | \\ \times \end{array}$$

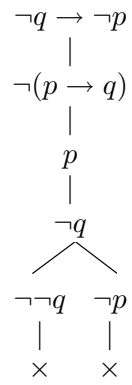
5.15 $\vdash_t \neg(p \wedge \neg p)$

$$\begin{array}{c} \neg\neg(p \wedge \neg p) \\ | \\ p \wedge \neg p \\ | \\ p \\ | \\ \neg p \\ | \\ \times \end{array}$$

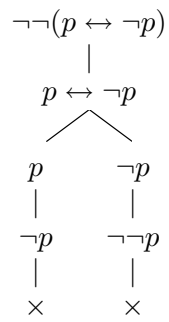
5.16 $p \rightarrow q \vdash_t \neg q \rightarrow \neg p$

$$\begin{array}{c} p \rightarrow q \\ | \\ \neg(\neg q \rightarrow \neg p) \\ | \\ \neg q \\ | \\ \neg\neg p \\ \wedge \\ \begin{array}{cc} \neg p & q \\ | & | \\ \times & \times \end{array} \end{array}$$

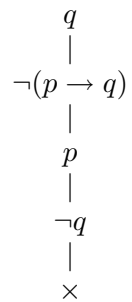
5.17 $\neg q \rightarrow \neg p \vdash_t p \rightarrow q$



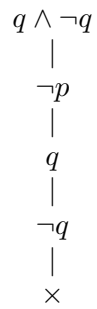
5.18 $\vdash_t \neg(p \leftrightarrow \neg p)$



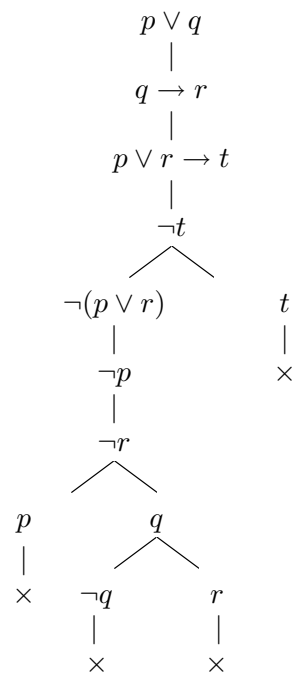
5.19 $q \vdash_t p \rightarrow q$



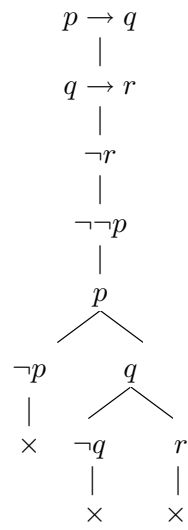
5.20 $q \wedge \neg q \vdash_t p$



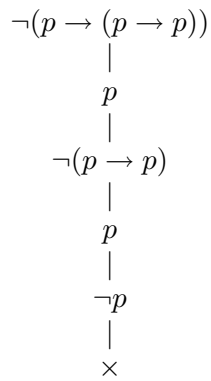
5.21 $p \vee q, q \rightarrow r, p \vee r \rightarrow t \vdash_t t$



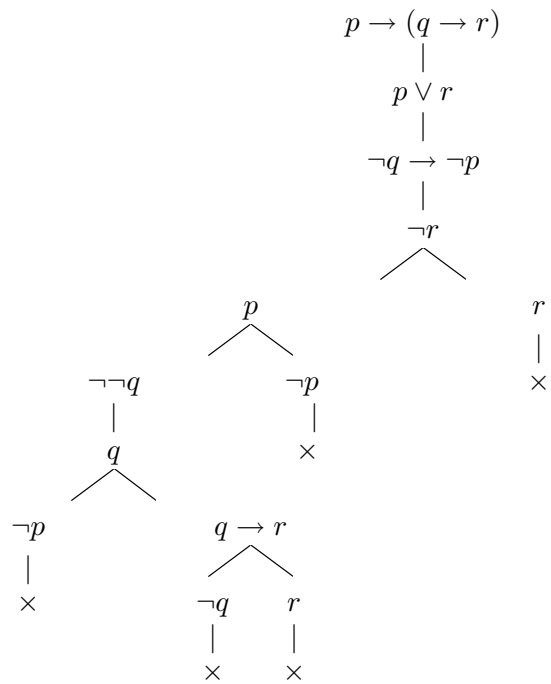
5.22 $p \rightarrow q, q \rightarrow r, \neg r \vdash_t \neg p$



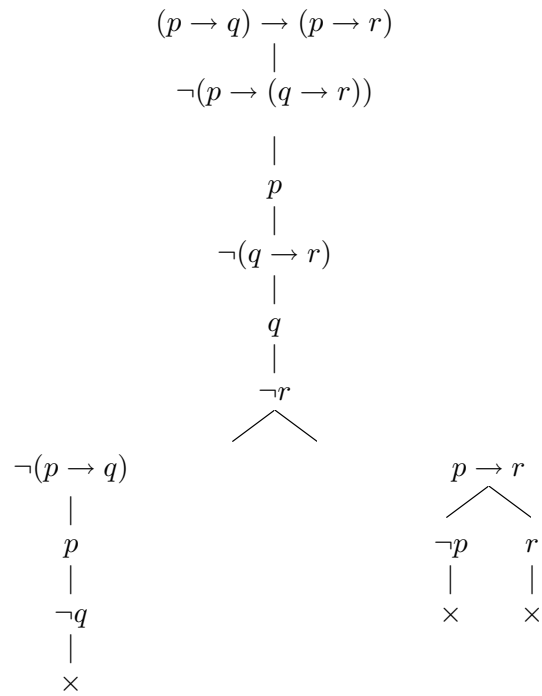
5.23 $\vdash_t p \rightarrow (p \rightarrow p)$



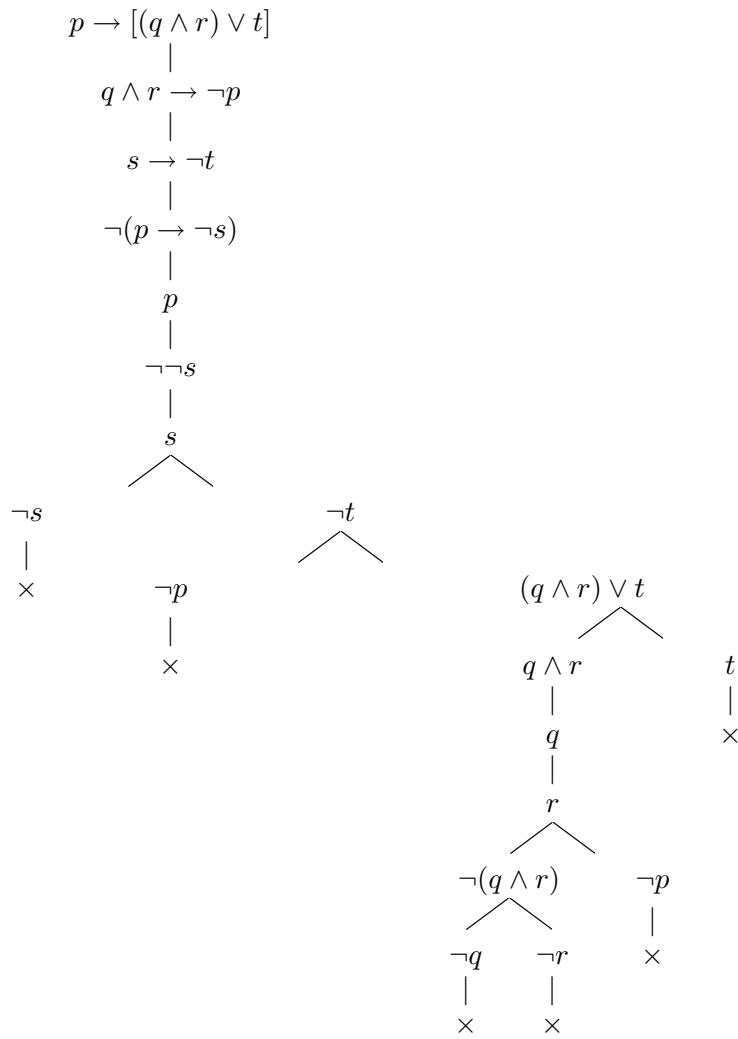
5.24 $p \rightarrow (q \rightarrow r), p \vee r, \neg q \rightarrow \neg p \vdash_t r$



5.25 $(p \rightarrow q) \rightarrow (p \rightarrow r) \vdash_t p \rightarrow (q \rightarrow r)$



5.26 $p \rightarrow [(q \wedge r) \vee t], q \wedge r \rightarrow \neg p, s \rightarrow \neg t \vdash_t p \rightarrow \neg s$



5.27 $p \rightarrow q \not\vdash_t q \rightarrow p$

$$\begin{array}{c} p \rightarrow q \\ | \\ \neg(q \rightarrow p) \\ | \\ q \\ | \\ \neg p \end{array}$$

$$\neg p \quad q \\ t(\neg p) = 1, t(q) = 1 \Rightarrow t(p) = 0, t(q) = 1$$

5.28 $\not\vdash_t p \leftrightarrow p \wedge q$

$$\neg(p \leftrightarrow (p \wedge q))$$

$$\begin{array}{cc} \neg p & p \\ | & | \\ p \wedge q & \neg(p \wedge q) \\ | & \\ p & \neg p \quad \neg q \\ & | \\ & \times \\ & | \\ & q \\ & | \\ & \times \end{array}$$

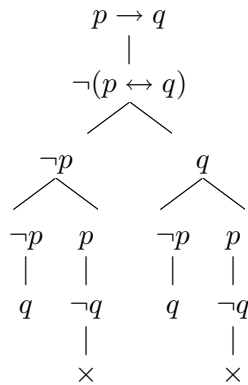
$$t(p) = 1, t(\neg q) = 1 \Rightarrow t(p) = 1, t(q) = 0$$

5.29 $p \rightarrow q \not\vdash_t \neg p \rightarrow \neg q$

$$\begin{array}{c} p \rightarrow q \\ | \\ \neg(\neg p \rightarrow \neg q) \\ | \\ \neg p \\ | \\ \neg\neg q \end{array}$$

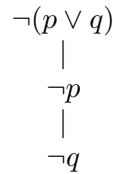
$$\neg p \quad q \\ t(\neg p) = 1, t(q) = 1 \Rightarrow t(\neg\neg q) = 1 \Rightarrow t(p) = 0, t(q) = 1$$

5.30 $p \rightarrow q \not\vdash_t p \leftrightarrow q$



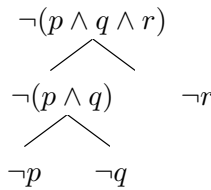
$$t(\neg p) = 1, t(q) = 1 \Rightarrow t(p) = 0, t(q) = 1$$

5.31 $\not\vdash_t p \vee q$



$$t(\neg p) = 1, t(\neg q) = 1 \Rightarrow t(p) = 0, t(q) = 0$$

5.32 $\not\vdash_t p \wedge q \wedge r$



Countermodels to $p \wedge q \wedge r$ are thereby such that $t(\neg p \vee \neg q \vee \neg r) = 1$. This means that there are 7 interpretations that can serve as counterexamples - namely all interpretations except $t(p) = 1, t(q) = 1, t(r) = 1$.

5.33 $\neg p \wedge \neg q$

5.34 $\neg p \vee q \vee r$

5.35 $p \wedge (q \vee r)$

5.36 $\neg p \vee q$

5.37 $\neg p \vee p$

5.38 $\neg p \vee p$

5.39 $q \vee \neg p$

5.40 $\neg p \vee p$

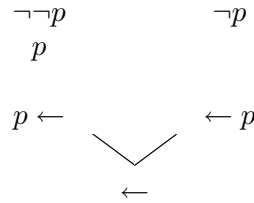
5.41 $\neg p \vee p$

5.42 $\neg p \vee \neg q \vee r$

5.43 $(p \vee \neg p \vee r) \wedge (\neg q \vee \neg p \vee r)$

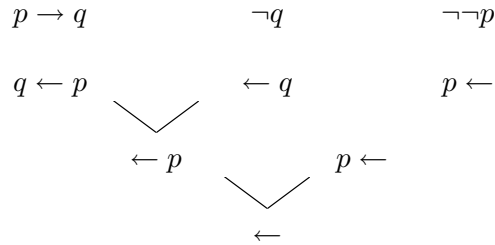
5.44 $(\neg p \vee q \vee t) \wedge (\neg p \vee r \vee t)$

5.45 $\neg\neg p \vdash_{res} p$

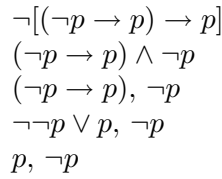


5.46 See the previous exercise.

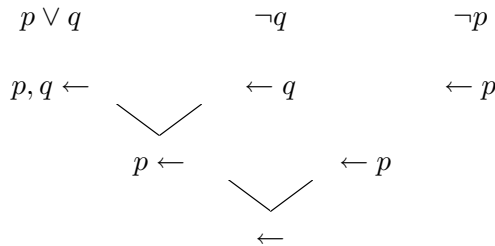
5.47 $p \rightarrow q, \neg q \vdash_{res} \neg p$



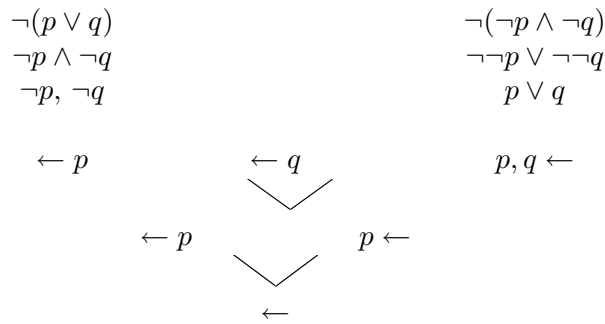
5.48 $\vdash_{res} (\neg p \rightarrow p) \rightarrow p$



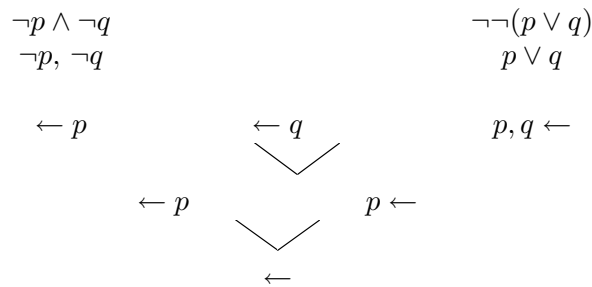
5.49 $p \vee q, \neg q \vdash_{res} p$



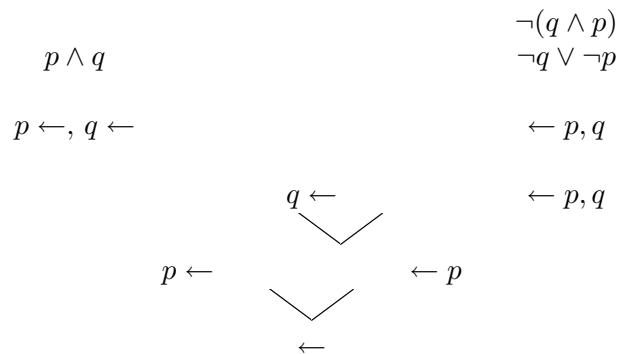
5.50 $\neg(p \vee q) \vdash_{res} \neg p \wedge \neg q$



5.51 $\neg p \wedge \neg q \vdash_{res} \neg(p \vee q)$

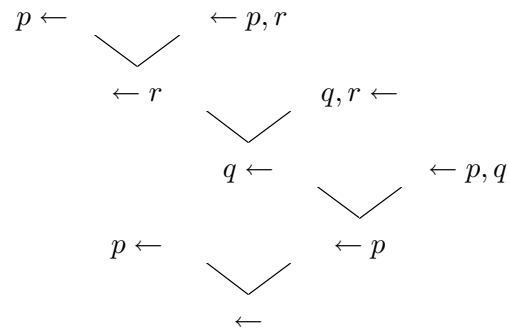


5.52 $p \wedge q \vdash_{res} q \wedge p$



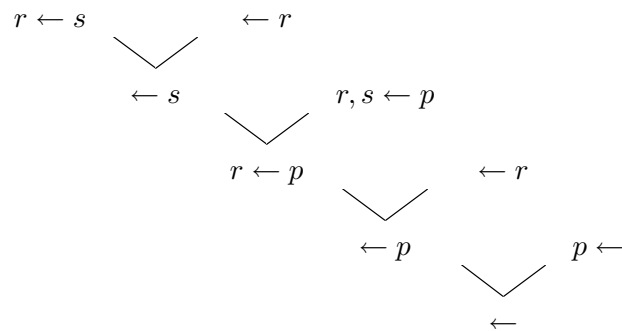
5.53 $p \wedge (q \vee r) \vdash_{res} (p \wedge q) \vee (p \wedge r)$

$$\begin{array}{ll}
 p \wedge (q \vee r) & \neg[(p \wedge q) \vee (p \wedge r)] \\
 p, q \vee r & \neg(p \wedge q) \wedge \neg(p \wedge r) \\
 p \leftarrow, q, r \leftarrow & \neg(p \wedge q), \neg(p \wedge r) \\
 & \neg p \vee \neg q \quad \neg p \vee \neg r \\
 & \leftarrow p, q \quad \leftarrow p, r
 \end{array}$$

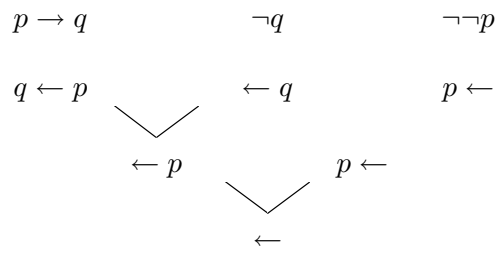


5.54 $p \rightarrow (r \vee s), s \rightarrow r, p \vdash_{res} r$

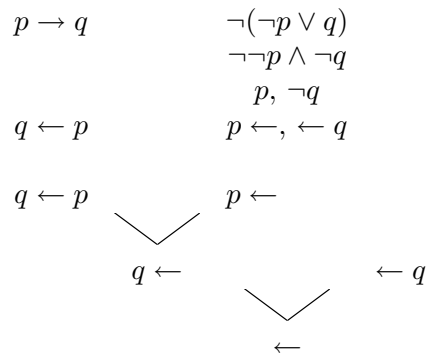
$$\begin{array}{lll}
 p \rightarrow (r \vee s) & s \rightarrow r & p \\
 \neg p \vee (r \vee s) & \neg s \vee r & \neg r \\
 \neg p \vee r \vee s & & \\
 r, s \leftarrow p & r \leftarrow s & p \leftarrow \leftarrow r
 \end{array}$$



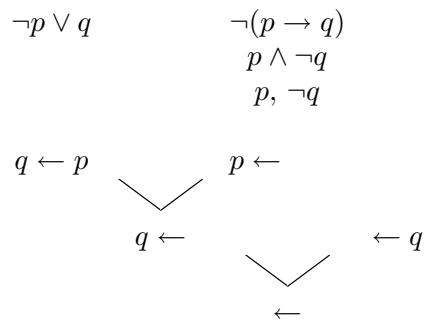
5.55 $p \rightarrow q, \neg q \vdash_{res} \neg p$



5.56 $p \rightarrow q \vdash_{res} \neg p \vee q$



5.57 $\neg p \vee q \vdash_{res} p \rightarrow q$



5.58 $\vdash_{res} p \vee \neg p$

$$\begin{array}{c} \neg(p \vee \neg p) \\ \neg p \wedge \neg\neg p \\ \neg p \qquad p \\ \leftarrow p \qquad p \leftarrow \\ \swarrow \quad \searrow \\ \leftarrow \end{array}$$

5.59 $\vdash_{res} \neg(p \wedge \neg p)$

$$\begin{array}{c} \neg\neg(p \wedge \neg p) \\ p \wedge \neg p \\ p \qquad \neg p \\ p \leftarrow \qquad \leftarrow p \\ \swarrow \quad \searrow \\ \leftarrow \end{array}$$

5.60 $p \rightarrow q \vdash_{res} \neg q \rightarrow \neg p$

$$\begin{array}{c} p \rightarrow q \quad \neg(\neg q \rightarrow \neg p) \\ \neg q \wedge \neg\neg p \\ \neg q, \quad \neg\neg p \\ q \leftarrow p \quad \leftarrow q \\ \swarrow \quad \searrow \\ \leftarrow p \quad p \leftarrow \\ \swarrow \quad \searrow \\ \leftarrow \end{array}$$

5.61 $\neg q \rightarrow \neg p \vdash_{res} p \rightarrow q$

$$\begin{array}{c} \neg q \rightarrow \neg p \quad \neg(p \rightarrow q) \\ \neg\neg q \vee \neg p \quad p \wedge \neg q \\ q \vee \neg p \quad \neg q, \quad p \\ q \leftarrow p \quad \leftarrow q \\ \swarrow \quad \searrow \\ \leftarrow p \quad p \leftarrow \\ \swarrow \quad \searrow \\ \leftarrow \end{array}$$

5.62 $\vdash_{res} \neg(p \leftrightarrow \neg p)$

$$\begin{array}{c} \neg\neg(p \leftrightarrow \neg p) \\ p \leftrightarrow \neg p \\ (\neg p \vee \neg\neg p) \wedge (\neg\neg p \vee p) \\ \neg p, \quad p \\ \leftarrow p \quad \quad p \leftarrow \\ \quad \quad \quad \swarrow \quad \searrow \\ \quad \quad \quad \leftarrow \end{array}$$

5.63 $q \vdash_{res} p \rightarrow q$

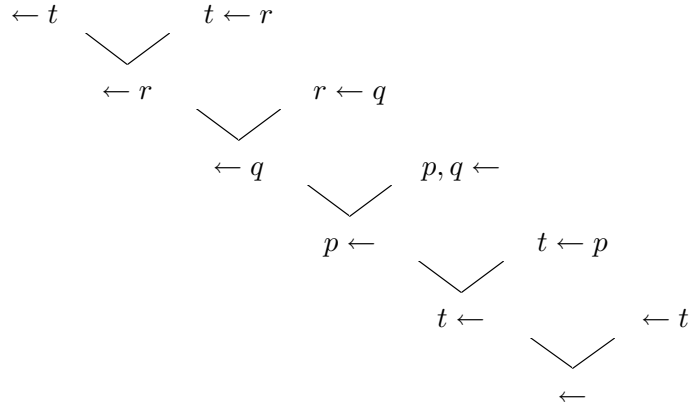
$$\begin{array}{c} \neg(p \rightarrow q) \\ q \quad \quad p \wedge \neg q \\ q \leftarrow \quad p \leftarrow, \leftarrow q \\ \leftarrow p \quad \quad \leftarrow q \\ \quad \quad \quad \swarrow \quad \searrow \\ \quad \quad \quad \leftarrow \end{array}$$

5.64 $q \wedge \neg q \vdash_{res} p$

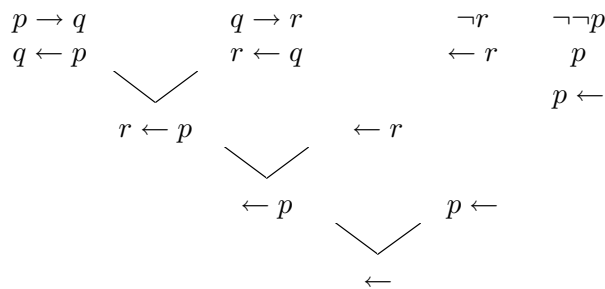
$$\begin{array}{c} q \wedge \neg q \quad \neg p \\ q \leftarrow, \leftarrow q \quad \leftarrow p \\ \leftarrow p \quad \quad \leftarrow q \\ \quad \quad \quad \swarrow \quad \searrow \\ \quad \quad \quad \leftarrow \end{array}$$

5.65 $p \vee q, q \rightarrow r, p \vee r \rightarrow t \vdash_{res} t$

$$\begin{array}{l}
 p \vee q \quad q \rightarrow r \quad p \vee r \rightarrow t \quad \neg t \\
 p, q \leftarrow \quad r \leftarrow q \quad \neg(p \vee r) \vee t \quad \leftarrow t \\
 \quad \quad \quad (\neg p \wedge \neg r) \vee t \\
 \quad \quad \quad (\neg p \vee t) \wedge (\neg r \vee t) \\
 \quad \quad \quad \neg p \vee t, \quad \neg r \vee t \\
 \quad \quad \quad t \leftarrow p, \quad t \leftarrow r
 \end{array}$$



5.66 $p \rightarrow q, q \rightarrow r, \neg r \vdash_{res} \neg p$

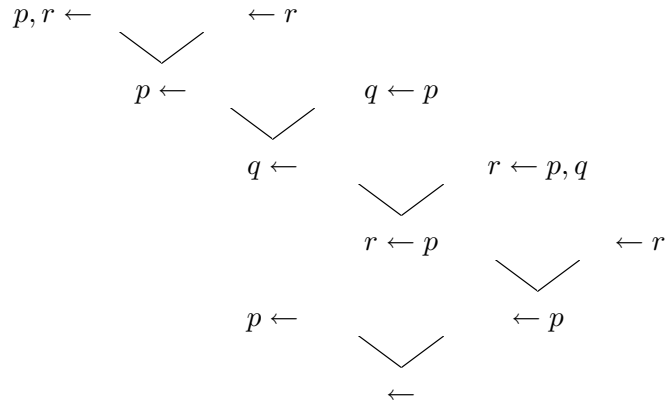


5.67 $\vdash_{res} p \rightarrow (p \rightarrow p)$



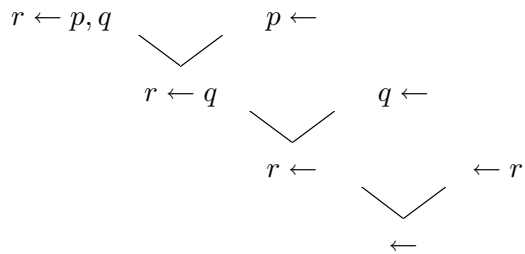
5.68 $p \rightarrow (q \rightarrow r), p \vee r, \neg q \rightarrow \neg p \vdash_{res} r$

$$\begin{array}{llll}
 p \rightarrow (q \rightarrow r) & p \vee r & \neg q \rightarrow \neg p & \neg r \\
 \neg p \vee (q \rightarrow r) & p, r \leftarrow & \neg \neg q \vee \neg p & \leftarrow r \\
 \neg p \vee (\neg q \vee r) & & q \vee \neg p & \\
 \neg p \vee \neg q \vee r & & q \leftarrow p & \\
 r \leftarrow p, q & & &
 \end{array}$$



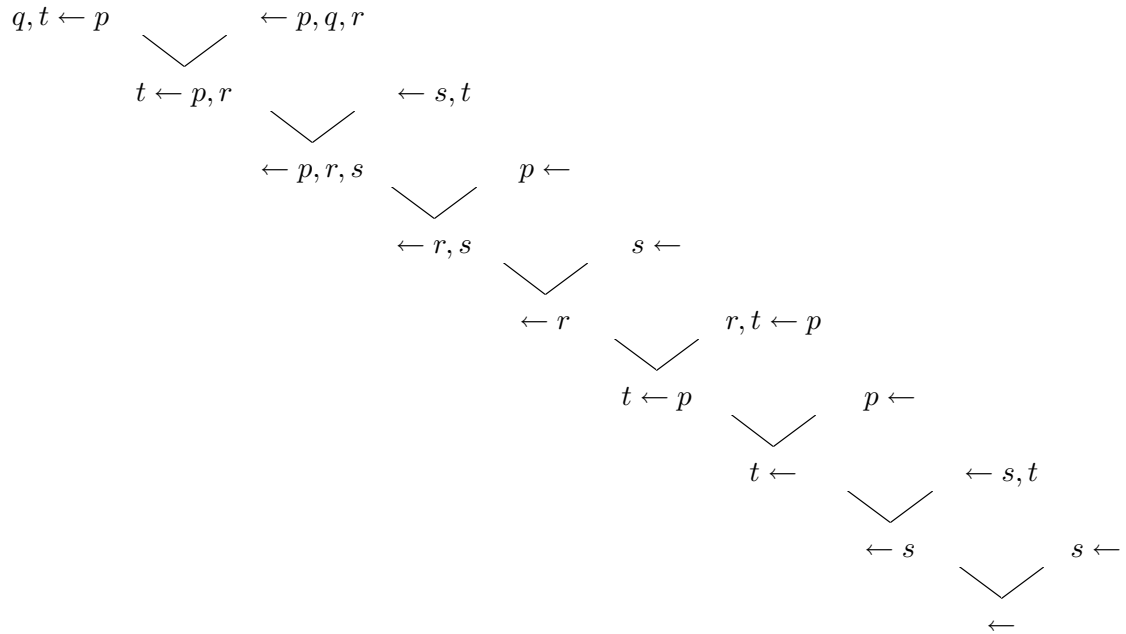
5.69 $(p \rightarrow q) \rightarrow (p \rightarrow r) \vdash_{res} p \rightarrow (q \rightarrow r)$

$$\begin{array}{ll}
 (p \rightarrow q) \rightarrow (p \rightarrow r) & \neg(p \rightarrow (q \rightarrow r)) \\
 \neg(p \rightarrow q) \vee (p \rightarrow r) & p \wedge \neg(q \rightarrow r) \\
 (p \wedge \neg q) \vee (\neg p \vee r) & p \wedge q \wedge \neg r \\
 (p \vee \neg p \vee r) \wedge (\neg q \vee \neg p \vee r) & p, q, \neg r \\
 p \vee \neg p \vee r, \neg q \vee \neg p \vee r & \\
 p, r \leftarrow p & r \leftarrow p, q \\
 r \leftarrow p, q & p \leftarrow, q \leftarrow, \leftarrow r
 \end{array}$$



5.70 $p \rightarrow ((q \wedge r) \vee t), q \wedge r \rightarrow \neg p, s \rightarrow \neg t \vdash_{res} p \rightarrow \neg s$

$$\begin{array}{llll}
 p \rightarrow ((q \wedge r) \vee t) & q \wedge r \rightarrow \neg p & s \rightarrow \neg t & \neg(p \rightarrow \neg s) \\
 \neg p \vee ((q \wedge r) \vee t) & \neg(q \wedge r) \vee \neg p & \neg s \vee \neg t & p \wedge \neg \neg s \\
 \neg p \vee ((q \vee t) \wedge (r \vee t)) & \neg q \vee \neg r \vee \neg p & & p, s \\
 (\neg p \vee q \vee t) \wedge (\neg p \vee r \vee t) & & & \\
 q, t \leftarrow p, & r, t \leftarrow p & \leftarrow p, q, r & \leftarrow s, t \quad p \leftarrow \quad s \leftarrow
 \end{array}$$



5.71 $\neg\neg p \vdash_{nd} p$

1. $\neg\neg p$ (p)
2. $\neg p$ (ant.)
3. \perp (1, 2, \perp - I)
4. p (2-3, \neg - E)

5.72 $p \vdash_{nd} \neg\neg p$

1. p (p)
2. $\neg p$ (ant.)
3. \perp (1, 2, \perp - I)
4. $\neg\neg p$ (2-3, \neg - I)

5.73 $p \rightarrow q, \neg q \vdash_{nd} \neg p$

1. $p \rightarrow q$ (p)
2. $\neg q$ (p)
3. p (ant.)
4. q (1, 3, \rightarrow - E)
5. \perp (2, 4, \perp - I)
6. $\neg p$ (3-5, \neg - I)

5.74 $\vdash_{nd} (\neg p \rightarrow p) \rightarrow p$

1. $(\neg p \rightarrow p)$ (ant.)
2. $\neg p$ (ant.)
3. p (1, 2, \rightarrow - E)
4. \perp (2, 3, \perp - I)
5. p (2-4, \neg - E)
6. $(\neg p \rightarrow p) \rightarrow p$ (1-5, \rightarrow - I)

5.75 $p \vee q, \neg q \vdash_{nd} p$

1. $p \vee q$ (p)
2. $\neg q$ (p)
3. $\neg p$ (ant.)
4. p (ant.)
5. \perp (3, 4, \perp - I)
6. q (ant.)
7. \perp (2, 6, \perp - I)
8. \perp (1, 4-5, 6-7, \vee - E)
9. p (3-8, \neg - E)

5.76 $\neg(p \vee q) \vdash_{nd} \neg p \wedge \neg q$

1. $\neg(p \vee q)$ (p)
2. p (ant.)
3. $p \vee q$ (2, $\vee - I$)
4. \perp (1, 3, $\perp - I$)
5. $\neg p$ (2-4, $\neg - I$)
6. q (ant.)
7. $p \vee q$ (6, $\vee - I$)
8. \perp (1, 7, $\perp - I$)
9. $\neg q$ (6-8, $\neg - I$)
10. $\neg p \wedge \neg q$ (5, 9, $\wedge - I$)

5.77 $\neg p \wedge \neg q \vdash_{nd} \neg(p \vee q)$

1. $\neg p \wedge \neg q$ (p)
2. $\neg p$ (1, $\wedge - E$)
3. $\neg q$ (1, $\wedge - E$)
4. $p \vee q$ (ant.)
5. p (ant.)
6. \perp (2, 5, $\perp - I$)
7. q (ant.)
8. \perp (3, 7, $\perp - I$)
9. \perp (4, 5-6, 7-8, $\vee - E$)
10. $\neg(p \vee q)$ (4-9, $\neg - I$)

5.78 $p \wedge q \vdash_{nd} q \wedge p$

1. $p \wedge q$ (p)
2. p (1, $\wedge - E$)
3. q (1, $\wedge - E$)
4. $q \wedge p$ (2, 3, $\wedge - I$)

5.79 $p \wedge (q \vee r) \vdash_{nd} (p \wedge q) \vee (p \wedge r)$

1. $p \wedge (q \vee r)$ (p)
2. $\neg((p \wedge q) \vee (p \wedge r))$ (a.)
3. $\neg(p \wedge q) \wedge \neg(p \wedge r)$ (2, exercise 5.76)
4. $\neg(p \wedge q)$ (3, $\wedge - E$)
5. $\neg(p \wedge r)$ (3, $\wedge - E$)
6. $\neg p \vee \neg q$ (4, example 5.38)
7. $\neg p \vee \neg r$ (5, example 5.38)
8. p (1, $\wedge - E$)
9. $q \vee r$ (1, $\wedge - E$)
10. $\neg p$ (a.)
11. \perp (8, 10, $\perp - I$)
12. $\neg q$ (a.)
13. r (9, 12, exercise 5.75)
14. $\neg\neg r$ (13, exercise 5.72)
15. $\neg p$ (7, 14, exercise 5.75)
16. \perp (8, 15, $\perp - I$)
17. \perp (6, 10–11, 12–16, $\vee - E$)
18. $(p \wedge q) \vee (p \wedge r)$ (2–17, $\neg - E$)

5.80 $p \vee p \vdash_{nd} p$

1. $p \vee p$ (p)
2. p (a.)
3. p (2, \vee)
4. p (1, 2–3, $\vee - E$)

5.81 $p \rightarrow q, \neg q \vdash_{nd} \neg p$

1. $p \rightarrow q$ (p)
2. $\neg q$ (p)
3. p (a.)
4. q (1, 3, $\rightarrow - E$)
5. \perp (2, 4, $\perp - I$)
6. $\neg p$ (3–5, $\neg - I$)

5.82 $p \rightarrow q \vdash_{nd} \neg p \vee q$

1. $p \rightarrow q$ (p)
2. $\neg(\neg p \vee q)$ (ant.)
3. $\neg\neg p \wedge \neg q$ (2, exercise 5.76)
4. $\neg\neg p$ (3, $\wedge - E$)
5. $\neg q$ (3, $\wedge - E$)
6. p (4, exercise 5.71)
7. q (1, 6, $\rightarrow - E$)
8. \perp (5, 7, $\perp - I$)
9. $\neg p \vee q$ (2-8, $\neg - E$)

5.83 $\neg p \vee q \vdash_{nd} p \rightarrow q$

1. $\neg p \vee q$ (p)
2. p (a.)
3. $\neg q$ (a.)
4. $\neg p$ (1, 3, exercise 5.75)
5. \perp (2, 4, $\perp - I$)
6. q (3-5, $\neg - E$)
7. $p \rightarrow q$ (2-6, $\rightarrow - E$)

5.84 $\vdash_{nd} p \vee \neg p$

1. $\neg(p \vee \neg p)$ (a.)
2. $\neg p \wedge \neg\neg p$ (1, exercise 5.76)
3. $\neg p$ (2, $\wedge - E$)
4. $\neg\neg p$ (3, $\wedge - E$)
5. \perp (3, 4, $\perp - I$)
6. $p \vee \neg p$ (1-5, $\neg - E$)

5.85 $\vdash_{nd} \neg(p \wedge \neg p)$

1. $p \wedge \neg p$ (a.)
2. p (1, $\wedge - E$)
3. $\neg p$ (1, $\wedge - E$)
4. \perp (2, 3, $\perp - I$)
5. $\neg(p \wedge \neg p)$ (1-4, $\neg - I$)

5.86 $p \rightarrow q \vdash_{nd} \neg q \rightarrow \neg p$

1. $p \rightarrow q$ (p)
2. $\neg q$ (a.)
3. p (a.)
4. q (1, 3, $\rightarrow - E$)
5. \perp (2, 4, $\perp - I$)
6. $\neg p$ (3-5, $\neg - I$)
7. $\neg q \rightarrow \neg p$ (2-6, $\rightarrow - I$)

5.87 $\neg q \rightarrow \neg p \vdash_{nd} p \rightarrow q$

1. $\neg q \rightarrow \neg p$ (p)
2. p (a.)
3. $\neg q$ (a.)
4. $\neg p$ (1, 3, $\rightarrow - E$)
5. \perp (2, 4, $\perp - I$)
6. q (3-5, $\neg - E$)
7. $p \rightarrow q$ (2-6, $\rightarrow - I$)

5.88 $\vdash_{nd} \neg(p \leftrightarrow \neg p)$

1. $p \leftrightarrow \neg p$ (a.)
2. $p \rightarrow \neg p$ (1, $\leftrightarrow - E$)
3. $\neg p \rightarrow p$ (1, $\leftrightarrow - E$)
4. $\neg p \vee \neg p$ (2, exercise 5.82)
5. $\neg\neg p \vee p$ (3, exercise 5.82)
6. $\neg p$ (4, exercise 5.80)
7. $\neg\neg p$ (a.)
8. \perp (6, 7, $\perp - I$)
9. p (a.)
10. \perp (6, 9, $\perp - I$)
11. \perp (5, 7-8, 9-10, $\vee - E$)
12. $\neg(p \leftrightarrow \neg p)$ (1-11, $\neg - I$)

5.89 $q \vdash_{nd} p \rightarrow q$

1. q (p)
2. p (a.)
3. q (1, u)
4. $p \rightarrow q$ (2-3, $\rightarrow - I$)

5.90 $q \wedge \neg q \vdash_{nd} p$

1. $q \wedge \neg q$ (p)
2. q (1, $\wedge - E$)
3. $\neg q$ (1, $\wedge - E$)
4. $\neg p$ (a.)
5. \perp (2, 3, $\perp - I$)
6. p (4-5, $\neg - E$)

5.91 $p \vee q, q \rightarrow r, p \vee r \rightarrow t \vdash_{nd} t$

1. $p \vee q$ (p)
2. $q \rightarrow r$ (p)
3. $p \vee r \rightarrow t$ (p)
4. p (a.)
5. $p \vee r$ (4, $\vee - I$)
6. t (3, 5, $\rightarrow - E$)
7. q (a.)
8. r (2, 7, $\rightarrow - E$)
9. $p \vee r$ (8, $\vee - I$)
10. t (3, 9, $\rightarrow - E$)
11. t (1, 4-6, 7-10, $\vee - E$)

5.92 $p \rightarrow q, q \rightarrow r, \neg r \vdash_{nd} \neg p$

1. $p \rightarrow q$ (p)
2. $q \rightarrow r$ (p)
3. $\neg r$ (p)
4. p (a.)
5. q (1, 4, $\rightarrow - E$)
6. r (2, 5, $\rightarrow - E$)
7. \perp (3, 6, $\perp - I$)
8. $\neg p$ (4-7, $\neg - I$)

5.93 $\vdash_{nd} p \rightarrow (p \rightarrow p)$

1. p (a.)
2. p (a.)
3. p (2, u)
4. $p \rightarrow p$ (2-3, $\rightarrow - I$)
5. $p \rightarrow (p \rightarrow p)$ (1-4, $\rightarrow - I$)

5.94 $p \rightarrow (q \rightarrow r), p \vee r, \neg q \rightarrow \neg p \vdash_{nd} r$

1. $p \rightarrow (q \rightarrow r)$ (p)
2. $p \vee r$ (p)
3. $\neg q \rightarrow \neg p$ (p)
4. p (a.)
5. $q \rightarrow r$ (1, 4, $\rightarrow - E$)
6. $\neg q$ (a.)
7. $\neg p$ (3, 6, $\rightarrow - E$)
8. \perp (4, 7, $\perp - I$)
9. q (6-8, $\neg - E$)
10. r (5, 9, $\rightarrow - E$)
11. r (a.)
12. r (11, u)
13. r (2, 4-10, 11-12, $\vee - E$)

5.95 $(p \rightarrow q) \rightarrow (p \rightarrow r) \vdash_{nd} p \rightarrow (q \rightarrow r)$

1. $(p \rightarrow q) \rightarrow (p \rightarrow r)$ (p)
2. p (a.)
3. q (a.)
4. p (a.)
5. q (3, u)
6. $p \rightarrow q$ (4-5, $\rightarrow - I$)
7. $p \rightarrow r$ (1, 6, $\rightarrow - E$)
8. r (2, 7, $\rightarrow - E$)
9. $q \rightarrow r$ (3-8, $\rightarrow - I$)
10. $p \rightarrow (q \rightarrow r)$ (2-9, $\rightarrow - I$)

5.96 $p \rightarrow [(q \wedge r) \vee t], q \wedge r \rightarrow \neg p, s \rightarrow \neg t \vdash_{nd} p \rightarrow \neg s$

1. $p \rightarrow [(q \wedge r) \vee t]$ (p)
2. $q \wedge r \rightarrow \neg p$ (p)
3. $s \rightarrow \neg t$ (p)
4. p (a.)
5. s (a.)
6. $(q \wedge r) \vee t$ (1, 4, $\rightarrow - E$)
7. $q \wedge r$ (a.)
8. $\neg p$ (2, 7, $\rightarrow - E$)
9. \perp (4, 7, $\perp - I$)
10. t (a.)
11. $\neg t$ (3, 5, $\rightarrow - E$)
12. \perp (10, 11, $\perp - I$)
13. \perp (6, 7–9, 10–12, $\vee - E$)
14. $\neg s$ (5–13, $\neg - I$)
15. $p \rightarrow \neg s$ (4–14, $\rightarrow - I$)